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LOCALIZATION OF SMALL INCLUSIONS FROM BOUNDARY MEASEUREMENTS: APPLICATION TO INCLUSIONS NEARLY TOUCHING THE BOUNDARY

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Abstract - The inverse problem of reconstructing, from boundary measurements, the position, the shape and the number of small inclusions in the interior of a domain, has received much attention over the last years (see the monograph by the two last authors for a comprehensive review). An important restriction for the solvability of the inverse problem was the assumption that the inclusions had to be sufficiently far from the boundary. This assumption was lifted in a recent work. The problem can be resumed as follows: we consider a homogeneous conducting object occupying a bounded domain in two or three dimensions, with a smooth connected boundary. The background potential is the solution to Laplace's equation with a Neumann boundary condition. A small inclusion, with positive conductivity different from the background, resides in the object and is nearly touching its boundary. We derive an explicit formula for the leading order boundary perturbations resulting from the small inclusion using layer potential techniques and polarization tensors. Careful numerical computations have enabled us to verify the validity of the asymptotic expansions. This means that a low cost imaging apparatus can be conceived on the basis of the mathematical analysis. That is, thanks to the analysis, the apparatus possesses very precise information regarding the actual localization and shape of the inclusions. Furthermore, there is no restriction concerning the proximity of the inclusions (for example, breast cancer tumours in a very early stage) to the boundary (the skin's surface).

1. INTRODUCTION

Consider a homogeneous conducting object occupying a bounded domain $\Omega \subset \mathbb{R}^2$, with a connected $\mathcal{C}^2$-boundary $\partial\Omega$. We will assume, for the sake of simplicity, that its conductivity is equal to 1. The background voltage potential, U, is the unique solution in $H^1(\Omega)$ to the boundary value problem

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega, \\ \dfrac{\partial U}{\partial \nu}\Big|_{\partial\Omega} = g, \ \displaystyle\int_{\partial\Omega} U = 0. \end{cases} \tag{1}$$

Here ν denotes the unit outward normal to the domain Ω and g represents the applied boundary current; it belongs to the set $L^2_0(\partial\Omega) = \{f \in L^2(\partial\Omega), \int_{\partial\Omega} f = 0\}$.

Consider a small inhomogeneity D inside Ω of conductivity equal to some positive constant $k \neq 1$ that is nearly touching the boundary $\partial\Omega$. We assume that

$$D = \epsilon B + z,$$

where $z \in \Omega$ is such that

$$\text{dist}(z, \partial\Omega) = M\epsilon.$$

Here B is a bounded domain in $\mathbb{R}^2$ containing the origin with a connected $\mathcal{C}^2$-boundary and the constant $M > \max_{x\in\partial B} |x|$.

The voltage potential in the presence of the conductivity inhomogeneity D is denoted u. It is the H^1-solution to

$$\begin{cases} \nabla \cdot \Big(1 + (k-1)\chi_D\Big)\nabla u = 0 \quad \text{in}\Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial\Omega} = g, \ \displaystyle\int_{\partial\Omega} u = 0, \end{cases} \tag{2}$$

where χ_D is the indicator function of D.

Our objective in this paper is to present an explicit formula for the leading order boundary perturbations resulting from the presence of the small conductivity inhomogeneity D. For the detailed derivation, the reader is referred to [4]. Our new asymptotic formula extends those already derived for a small inhomogeneity far away from the boundary [1] to the case of one nearly touching the boundary. If the conductivity inclusion is not too close to the boundary it can be modeled by a dipole. This approximation is valid when the field within the inclusion is nearly constant. On decreasing the inclusion-boundary separation, the assumption that the field within the inclusion is nearly constant begins to fail because higher order multi-poles become significant due to the inclusion-boundary interaction. Our new approximation which is valid when the inclusion is at a distance comparable to its diameter apart from the boundary provides some essential insight for understanding the inclusion-boundary interaction.

Asymptotic formulae for the boundary perturbations due to the presence of conductivity inhomogeneities are of significant interest from an imaging point of view. For instance: if one has a very detailed knowledge of the boundary signatures of conductivity inhomogeneities, then it becomes possible to design very effective algorithms to identify their location and certain properties of their shapes. We refer the reader to [2, 5, 6] for examples of numerical methods based on such specific formulae. Since our formula carries information on the location, the conductivity and the volume of the inclusion, it can be efficiently exploited for imaging inclusions close to the boundary.

Although we deal with the problem only in two-dimensional space, all results in this paper are valid for general dimension $d \geq 3$ with minor modifications.

2. ASYMPTOTIC EXPANSIONS

We first give a representation formula that serves to express the solution u to the conductivity problem (2) through its harmonic part H defined by

$$H(x) := \mathcal{D}_\Omega(u|_{\partial\Omega})(x) - \mathcal{S}_\Omega g(x) \quad \text{for } x \in \mathbb{R}^2 \setminus \partial\Omega \tag{3}$$

where $\mathcal{S}$ and $\mathcal{D}$ are the single and double layer potentials for the Laplacian operator. The following representation formula holds:

$$u(x) = H(x) + \mathcal{S}_D(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial H}{\partial \nu}|_{\partial D}) \quad \text{for } x \in \Omega, \tag{4}$$

where $\lambda = \frac{k+1}{2(k-1)}$.

Based on known results for layer potentials and a representation formula for the conductivity in terms of these potentials (see [4, 1] for details), we can show that for $x \in \partial\Omega$

$$\begin{aligned}(u-U)(x) &+ \int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1}(\nu \cdot \nabla \mathcal{D}_\Omega(u-U))(y)\, d\sigma_y \\ &= -\int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1}(\nu \cdot \nabla U)(y)\, d\sigma_y,\end{aligned} \tag{5}$$

where the singular integral operator, $\mathcal{K}_D$, is defined as

$$\mathcal{K}_D\phi(x) == \frac{1}{2\pi}\text{p.v.}\int_{\partial D} \frac{\langle \nu_y, y-x\rangle}{|x-y|^2}\phi(y)d\sigma_y$$

and $\mathcal{K}_D^*$ is the L^2-adjoint of $\mathcal{K}_D$. If $\mathcal{O}$ is a two dimensional disk with radius R, then for $x, y \in \partial\mathcal{O}$,

$$\frac{\langle \nu_x, x-y\rangle}{|x-y|^2} = \frac{1}{2R},$$

and hence

$$\mathcal{K}_\mathcal{O}^*\phi(x) = \mathcal{K}_\mathcal{O}\phi(x) = \frac{1}{4\pi R}\int_{\partial\mathcal{O}} \phi(y)\, d\sigma_y. \tag{6}$$

The formula (5) is a representation formula for the perturbations $u - U$ on $\partial\Omega$. The Neumann function $N(x, y)$ for Δ in Ω corresponding to a Dirac mass at z is defined as the solution to

$$\begin{cases} \Delta_x N(x,z) = -\delta_z & \text{in } \Omega, \\ \dfrac{\partial N}{\partial \nu}|_{\partial\Omega} = -\dfrac{1}{|\partial\Omega|}, \\ \displaystyle\int_{\partial\Omega} N(x,y)d\sigma_x = 0 & \text{for } y \in \Omega. \end{cases} \tag{7}$$

The following formula relates the fundamental solution Γ to the Neumann function.

$$(-\frac{1}{2}I + \mathcal{K}_\Omega)(N(\cdot, y))(x) = \Gamma(x-y) \quad \text{modulo constants}, \quad x \in \partial\Omega, y \in \Omega. \tag{8}$$

Observe that if Ω is the two-dimensional disk, then $N(x,y) = -2\Gamma(x-y)$ modulo constants, for $x \in \partial\Omega, y \in \Omega$.

We also have that

$$u(x) = U(x) - \int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1}(\frac{\partial H}{\partial \nu}|_{\partial D})(y)\, d\sigma_y, \quad \text{for } x \in \overline{\Omega}. \tag{9}$$

This representation formula provides us with an ideal tool for the rigorous derivation of the leading order boundary perturbations resulting from the presence of the conductivity inhomogeneity D.

For $v \in L^\infty(\partial\Omega)$, let

$$Tv(x) := \int_{\partial D} N(x,y)(\lambda I - \mathcal{K}_D^*)^{-1}(\nu \cdot \nabla \mathcal{D}_\Omega v)(y)\, d\sigma_y, \quad x \in \partial\Omega. \tag{10}$$

Since $(\lambda I - \mathcal{K}_D^*)^{-1}(\nu \cdot \nabla \mathcal{D}_\Omega v)$ has the mean value zero, we get

$$Tv(x) = \int_{\partial D} \frac{N(x,y) - N(x,z)}{\epsilon}(\lambda I - \mathcal{K}_D^*)^{-1}(\epsilon\nu \cdot \nabla \mathcal{D}_\Omega v)(y)\, d\sigma_y. \tag{11}$$

We view (5) as an integral equation

$$(I+T)(u-U) = F \quad \text{on } \partial\Omega \tag{12}$$

where the definition of F is obvious. We can show that if $\partial\Omega$ of class $\mathcal{C}^2$ then $I+T$ is invertible on $\mathcal{C}^0(\partial\Omega)$. We thus have that

$$u(x) - U(x) = (I+T)^{-1}(F)(x), \quad x \in \partial\Omega. \tag{13}$$

To derive an asymptotic expansion of $u-U$ on $\partial\Omega$ we now investigate the asymptotic behavior of the operator T as $\epsilon \to 0$.

Let

$$W(x) := \nu_{z_0} \cdot \int_{\partial D} \frac{N(x,y) - N(x,z)}{\epsilon}(\lambda I - \mathcal{K}_D^*)^{-1}(w\nu)(y)\, d\sigma_y, \quad x \in \partial\Omega. \tag{14}$$

Then a Lemma shows that for $f \in \mathcal{C}^0(\partial\Omega)$

$$Tf(x) = -W(x)f(z_0) + T_1 f(x), \tag{15}$$

where

$$T_1 f(x) = \int_{\partial D} \frac{N(x,y) - N(x,z)}{\epsilon}(\lambda I - \mathcal{K}_D^*)^{-1}((\epsilon\nabla\mathcal{D}_\Omega f + wf(z_0)\nu_{z_0}) \cdot \nu)(y)\, d\sigma_y.$$

We can then show that

$$\|T_1 f\|_{L^\infty(\partial\Omega)} \leq C\epsilon^{-1+1/p}\|\epsilon\nabla\mathcal{D}_\Omega f + wf(z_0)\nu_{z_0}\|_{L^q(\partial D)} \leq C(s_f(\sqrt{\epsilon}) + \sqrt{\epsilon})\|f\|_{L^\infty(\partial\Omega)}, \tag{16}$$

where $1/p + 1/q = 1$, $p, q > 1$ and

$$s_f(\epsilon) = \sup_{|x-z_0|\leq\epsilon} |f(x) - f(z_0)|.$$

Moreover one can show in a similar way that if x is far away from z_0, then

$$|T_1 f(x)| \leq C\epsilon(s_f(\sqrt{\epsilon}) + \sqrt{\epsilon})\|f\|_{L^\infty(\partial\Omega)}. \tag{17}$$

Let

$$M_W f := W(x)f(z_0), \quad f \in \mathcal{C}^0(\partial\Omega).$$

Then we get

$$I + T = I - M_W + T_1. \tag{18}$$

It is easy to see that $I - M_W$ is invertible provided that $W(z_0) \neq 1$. In fact, $(I - M_W)^{-1}$ is given by

$$(I - M_W)^{-1}(f)(x) = f(x) + \frac{W(x)}{1 - W(z_0)} f(z_0). \tag{19}$$

In view of (13), we get

$$u - U = (I - M_W)^{-1}(F) - (I + T)^{-1} T_1 (I - M_W)^{-1}(F) \quad \text{on } \partial\Omega. \tag{20}$$

We now investigate the asymptotic behavior of F as $\epsilon \to 0$. We suppose from now on that $g \in \mathcal{C}^1(\partial\Omega)$ and Ω is of class $\mathcal{C}^2$ so that $U \in \mathcal{C}^2(\overline{\Omega})$. We first observe that

$$\|F\|_{L^\infty(\partial\Omega)} \leq C\epsilon \|\nabla U\|_{L^\infty(\partial D)}.$$

This can be proved in a similar way as before.

Since $U \in \mathcal{C}^2(\overline{\Omega})$,

$$\nabla U|_{\partial D} = \nabla U(z_0) + O(\epsilon),$$

which gives after changes of variables

$$F(x) = -\epsilon \nabla U(z_0) \cdot \Big(\int_{\partial B} N(x, z + \epsilon y)(\lambda I - \mathcal{K}_B^*)^{-1}(\nu)\, d\sigma_y \Big) + O(\epsilon^2), \tag{21}$$

if x is close to z_0. Moreover, if x is far away from z_0 or $|x - z_0| \gg O(\epsilon)$, then

$$\begin{aligned} \int_{\partial B} N(x, z + \epsilon y)(\lambda I - \mathcal{K}_B^*)^{-1}(\nu)\, d\sigma_y &= \int_{\partial B} [N(x, z + \epsilon y) - N(x, z)](\lambda I - \mathcal{K}_B^*)^{-1}(\nu)\, d\sigma_y \\ &= \epsilon \nabla N(x, z) P + O(\epsilon^2) \\ &= \epsilon \nabla N(x, z_0) P + O(\epsilon^2), \end{aligned}$$

where $P = \int_{\partial B} y(\lambda I - \mathcal{K}_B^*)^{-1}(\nu)\, d\sigma_y$ is the polarization tensor. Thus in this case we obtain

$$F(x) = -\epsilon^2 \nabla U(z_0)^t P \nabla N(x, z_0) + O(\epsilon^3) \quad \text{if } |x - z_0| \gg O(\epsilon). \tag{22}$$

The reader is referred to [3, 6] for extensive studies on polarization tensors.

We claim that

$$s_F(\sqrt{\epsilon}) = \sup_{|x - z_0| \leq \sqrt{\epsilon}} |F(x) - F(z_0)| = O(\sqrt{\epsilon}).$$

In fact, since $\partial\Omega$ is $\mathcal{C}^2$, $|\nabla_x N(x, y)| \leq C|x - y|^{-1}$ for $x \in \partial\Omega$ and $y \in \Omega$. Therefore,

$$\begin{aligned} |F(x) - F(z_0)| &\leq \int_{\partial D} |N(z_0, y) - N(x, y)| |(\lambda I - \mathcal{K}_D^*)^{-1}(\nu \cdot \nabla U)(y)|\, d\sigma_y \\ &\leq \Big(\int_{\partial D} |N(z_0, y) - N(x, y)|^p d\sigma_y \Big)^{1/p} \|(\lambda I - \mathcal{K}_D^*)^{-1}(\nu \cdot \nabla U)\|_{L^q(\partial D)} \\ &\leq C\epsilon^{-1+1/p} \|\nabla U\|_{L^q(\partial D)} \\ &\leq C \|\nabla U\|_{L^\infty(\partial D)} |x - z_0|. \end{aligned}$$

We then obtain from (16) and (17) that

$$T_1 (I - M_W)^{-1} F(x) = \begin{cases} O(\epsilon^{3/2}) & \text{if } |x - z_0| = O(\epsilon), \\ O(\epsilon^{5/2}) & \text{if } |x - z_0| \gg O(\epsilon). \end{cases} \tag{23}$$

In a similar way one can see that

$$W(x) = \begin{cases} O(1) & \text{if } |x - z_0| = O(\epsilon), \\ O(\epsilon) & \text{if } |x - z_0| \gg O(\epsilon). \end{cases} \tag{24}$$

We finally obtain the following theorem from (13), (19), (21), (22), (23), and (24).

Theorem 1 *Suppose that $g \in \mathcal{C}^1(\partial\Omega)$ and Ω is of class $\mathcal{C}^2$. We also assume that $W(z_0) \neq 1$, where W is the function introduced in (14). Then the following asymptotic expansion holds uniformly on $\partial\Omega$:*

$$(u - U)(x) = -\epsilon \nabla U(z_0) \cdot \Big(\int_{\partial B} N(x, z + \epsilon y)(\lambda I - \mathcal{K}_B^*)^{-1}(\nu)\, d\sigma_y \Big) - \epsilon \frac{W(x)}{1 - W(z_0)} \nabla U(z_0) \cdot \Big(\int_{\partial B} N(z_0, z + \epsilon y)(\lambda I - \mathcal{K}_B^*)^{-1}(\nu)\, d\sigma_y \Big) + O(\epsilon^{3/2}).$$

Moreover, if $|x - z_0| \gg O(\epsilon)$ then

$$(u - U)(x) = -\epsilon^2 \nabla U(z_0)^t P \nabla N(x, z_0) - \epsilon \frac{W(x)}{1 - W(z_0)} \nabla U(z_0) \cdot \Big(\int_{\partial B} N(z_0, z + \epsilon y)(\lambda I - \mathcal{K}_B^*)^{-1}(\nu)\, d\sigma_y \Big) + O(\epsilon^{5/2}),$$

where $P = \int_{\partial B} y (\lambda I - \mathcal{K}_B^*)^{-1}(\nu)\, d\sigma_y$ is the polarization tensor and N is the Neumann function defined in (7).

Since $\int_{\partial B} N(x, z + \epsilon y)(\lambda I - \mathcal{K}_B^*)^{-1}(\nu)\, d\sigma_y = O(1)$ for x near z_0, Theorem 1 and (24) show that $(u - U)(x) = O(\epsilon)$ near z_0, while $(u - U)(x) = O(\epsilon^2)$ for x far away from z_0. Thus $u - U$ has a relative peak near z_0.

Some words are in order for the condition $W(z_0) \neq 1$. Since

$$|W(z_0)| \leq C \|w\|_{L^\infty(\partial D)},$$

where the constant C is independent of M, the condition is fulfilled if M is large enough.

3. NUMERICAL EXAMPLE OF A UNIT DISK CONTAINING A SINGLE DISK-SHAPED IMPERFECTION

In this section, we consider a unit disk in $\mathbb{R}^2$ with background conductivity 1 containing a single disk-shaped imperfection of small radius ϵ and conductivity k. The imperfection is centered at $z = (1 - M\epsilon, 0)$ on the axis $y = 0$ at a distance $(M - 1)\epsilon$ from the boundary where the constant $M > 1$. Let $z_0 = (1, 0)$.

Our aim is to examine the perturbation due to the presence of the imperfection on the Dirichlet boundary measurements as ϵ tends to zero. We thus examine the viability of our results in Theorem 1 by numerical examples.

Since

$$\int_{\partial D} \nu_y\, d\sigma_y = 0 \text{ and } \int_{\partial\Omega} N(x, y) d\sigma_x = 0, \text{ for } y \in \Omega,$$

then using property (6) we have

$$(\lambda I - \mathcal{K}_D^*)^{-1}(\nu)(y) = \frac{1}{\lambda} \nu_y, \forall\, y \in \partial\Omega,$$

and

$$N(x, y) = -2\Gamma(x - y) \text{ modulo constants}, \forall\, x \in \partial\Omega, y \in \Omega.$$

From Theorem 1 it then follows that

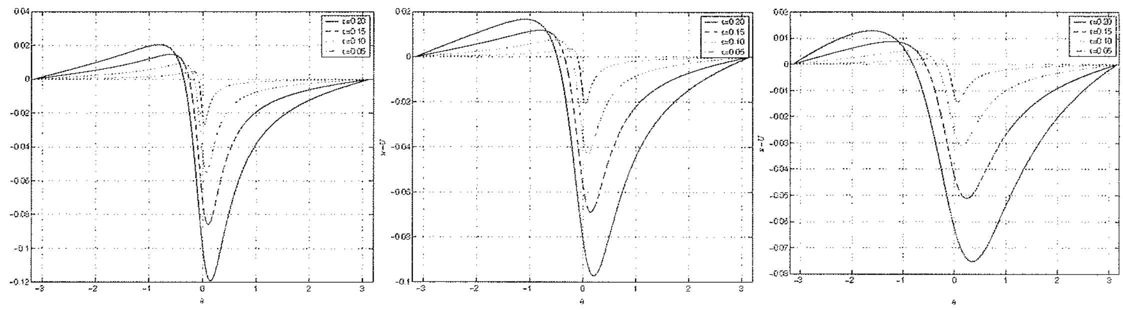
(i)

$$(u - U)(x) \simeq -\frac{\epsilon}{\pi\lambda} \nabla U(z_0) \cdot \Big(\int_{\partial B} \log|x - z - \epsilon y| \nu_y\, d\sigma_y \Big) - \frac{\epsilon W(x)}{\pi\lambda(1 - W(z_0))} \nabla U(z_0) \cdot \Big(\int_{\partial B} \log|z_0 - z - \epsilon y| \nu_y\, d\sigma_y \Big), \quad x \in \partial\Omega.$$

(ii)

$$(u - U)(x) \simeq -\frac{4\epsilon^2}{\lambda} \nabla U(z_0) \cdot \frac{x - z_0}{|x - z_0|^2} - \frac{\epsilon W(x)}{\pi\lambda(1 - W(z_0))} \nabla U(z_0) \cdot \Big(\int_{\partial B} \log|M\nu_{z_0} - y| \nu_y\, d\sigma_y \Big),$$

if $|x - z_0| \gg O(\epsilon)$.



(a) (b) (c)

Figure 1: Perturbation of boundary conductivity, $(u - U)|_{\partial\Omega}$, for $k = 2$ and ϵ varying with (a) $M = 1.5$, (b) $M = 2$ and (c) $M = 3$. Line styles: solid $\epsilon = 0.2$; dashed $\epsilon = 0.15$; dotted $\epsilon = 0.1$ and dot-dash $\epsilon = 0.05$.

(iii)

$$(u - U)(z_0) \simeq -\frac{\epsilon}{\pi\lambda(1 - W(z_0))} \nabla U(z_0) \cdot \Big(\int_{\partial B} \log|M\nu_{z_0} - y|\nu_y \, d\sigma_y \Big).$$

We now present numerical simulations using these asymptotic expansions. In these experiments, we examine numerically the transmission problem (2) in cylindrical coordinates (r, θ) with Neumann boundary data $g(1, \theta) = \cos\theta + \sin\theta$. The analytical solution of the homogeneous problem (1) is then given by $U(r, \theta) = r(\cos\theta + \sin\theta)$. Therefore, $(u - U)(z_0)$ can be approximated as follows:

$$(u - U)(z_0) \simeq -\frac{\epsilon(k-1)}{\pi(k+1)(1 - W(z_0))} \int_0^{2\pi} \log\left((M - \cos\theta)^2 + \sin^2\theta \right) (\cos\theta + \sin\theta) \, d\theta, \tag{25}$$

where

$$W(z_0) = \frac{1-k}{2\pi(k+1)} \int_0^{2\pi} \log\left(\frac{(M - \cos\theta)^2 + \sin^2\theta}{M^2} \right) \frac{\cos\theta}{M - \cos\theta} \, d\theta.$$

The first set of computations (see Figure 1) shows the dependence of the perturbation of the boundary conductivity $(u - U)|_{\partial\Omega}$ as a function of the distance variable M for different values of ϵ and for a fixed imperfection conductivity of $k = 2$. The next three figures (Figure 2) show the results for a larger value of the conductivity, $k = 10$. We observe that the minimal value (near $\theta = 0$) is constant and this is clearer as the distance M *decreases*. We can conclude that the perturbation amplitude is asymptotically *first order* in ϵ.

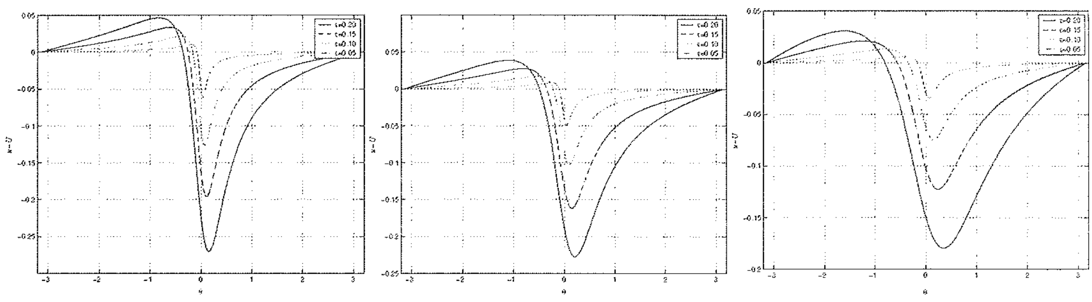
We can also plot these same results for $k = 10$ fixed as a function of M with $\epsilon = 0.1$ - see Figure 3 (a). We clearly observe the dependence of the amplitude and sharpness of the peak as a function of the distance.

The results of the above computations are resumed in Table 1 and Table 2 where the maximal value of $|u - U|$ on the boundary $\partial\Omega$ is given as a function of the three parameters k, M and ϵ. To check the influence of the angular position of the perturbation, a computation with $z = (0.5, 0.5)$ was performed. In Figure 4 we observe that the perturbation is indeed centered at $\theta = \pi/4$. We conclude that the *angular position* of the imperfection corresponds to the position of the perturbation peak.

Next, we compare in Table 3 the values of $(u - U)(z_0)$ computed from the asymptotic formula (25) with those computed by a direct simulation as in Tables 1 and 2.

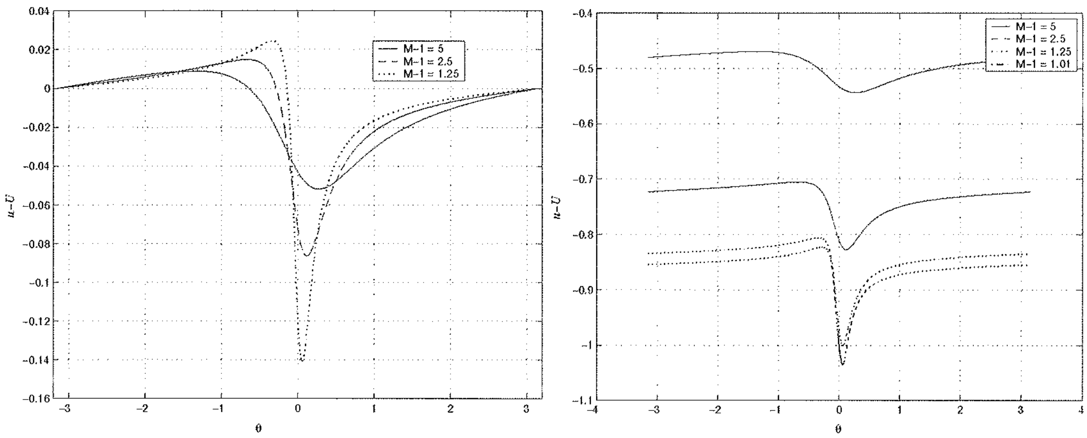
Finally, we consider a homogeneous disk with a perfectly conducting circular imperfection. The boundary condition on the perimeter of the imperfection is homogeneous Dirichlet, $u = 0$. The results as a function of M are shown in Figure 3 (b). As in the cases above,

- the peak of the perturbation corresponds to the angular position of the imperfection;
- as the imperfection approaches the boundary, the peak amplitude tends to a finite limit.



(a) (b) (c)

Figure 2: Perturbation of boundary conductivity, $(u-U)|_{\partial\Omega}$, for $k=10$ and ϵ varying with (a) $M=1.5$, (b) $M=2$ and (c) $M=3$. Line styles: solid $\epsilon=0.2$; dashed $\epsilon=0.15$; dotted $\epsilon=0.1$ and dot-dash $\epsilon=0.05$.



(a) (b)

Figure 3: Perturbation of boundary conductivity, $(u-U)|_{\partial\Omega}$, for $\epsilon=0.1$ and varying distance M with (a) $k=10$, (b) $k=+\infty$. Line styles: solid $M-1=5$; dashed $M-1=2.5$; dotted $M-1=1.25$ and dot-dash $M-1=1.01$.

	$M=1.5$	$M=2.0$	$M=3.0$
$\epsilon=0.20$	0.119 (**0.60**)	0.097 (**0.49**)	0.075 (**0.38**)
$\epsilon=0.15$	0.086 (**0.57**)	0.069 (**0.46**)	0.051 (**0.34**)
$\epsilon=0.10$	0.055 (**0.55**)	0.044 (**0.44**)	0.031 (**0.31**)
$\epsilon=0.05$	0.027 (**0.54**)	0.021 (**0.42**)	0.014 (**0.28**)

Table 1: $\max_{\partial\Omega}|u-U|$ and $\max_{\partial\Omega}|u-U|/\epsilon$ (in bold) for $k=2$.

	$M=1.5$	$M=2.0$	$M=3.0$
$\epsilon=0.20$	0.270 (**1.35**)	0.227 (**1.14**)	0.180 (**0.90**)
$\epsilon=0.15$	0.196 (**1.30**)	0.162 (**1.07**)	0.123 (**0.82**)
$\epsilon=0.10$	0.126 (**1.26**)	0.103 (**1.03**)	0.075 (**0.75**)
$\epsilon=0.05$	0.061 (**1.22**)	0.049 (**1.00**)	0.035 (**0.69**)

Table 2: $\max_{\partial\Omega}|u-U|$ and $\max_{\partial\Omega}|u-U|/\epsilon$ (in bold) for $k=10$.

	$M = 3.0$	$M = 4.0$	$M = 5.0$
$\epsilon = 0.05$	0.0118 (**0.236**)	0.0093 (**0.185**)	0.0076 (**0.152**)
	0.0116 (**0.232**)	0.0085 (**0.171**)	0.0068 (**0.135**)
$\epsilon = 0.02$	0.0046 (**0.228**)	0.0035 (**0.173**)	0.0028 (**0.140**)
	0.0046 (**0.232**)	0.0034 (**0.171**)	0.0027 (**0.135**)
$\epsilon = 0.01$	0.0023 (**0.230**)	0.0017 (**0.170**)	0.0014 (**0.135**)
	0.0023 (**0.232**)	0.0017 (**0.171**)	0.0014 (**0.135**)

Table 3: Comparison of $(u - U)(z_0)$ computed numerically (upper lines) and $(u - U)(z_0)$ computed from the asymptotic formula (25) (lower lines) for $k = 2$. Bold values are $(u - U)(z_0)/\epsilon$.

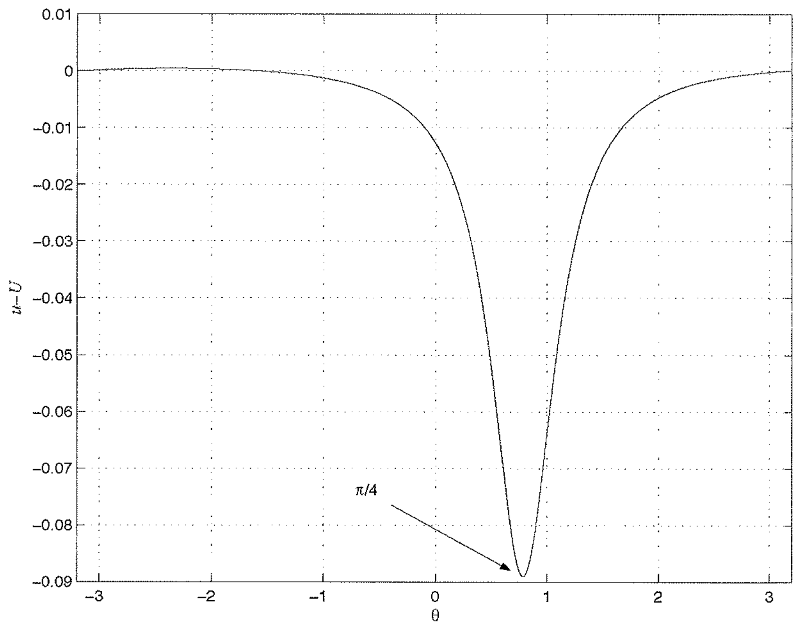


Figure 4: Perturbation of boundary conductivity, $(u - U)|_{\partial\Omega}$, for $z = (0.5, 0.5)$.

4. CONCLUSIONS

We have presented an asymptotic expansion that allows the accurate reconstruction of the position and shape of small imperfections that are close to the surface. The numerical experiments reveal that these asymptotic results are indeed attainable and could thus be implemented in new imaging protocols, especially for the early detection of breast cancer.

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